

CLASSICAL AND STRONGLY CLASSICAL 2-ABSORBING SECOND SUBMODULES

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ABSTRACT. In this paper, we will introduce the concept of classical (resp. strongly classical) 2-absorbing second submodules of modules over a commutative ring as a generalization of 2-absorbing (resp. strongly 2-absorbing) second submodules and investigate some basic properties of these classes of modules.

1. INTRODUCTION

Throughout this paper, R will denote a commutative ring with identity and “ \subset ” will denote the strict inclusion. Further, \mathbb{Z} will denote the ring of integers.

Let M be an R -module. A proper submodule P of M is said to be *prime* if for any $r \in R$ and $m \in M$ with $rm \in P$, we have $m \in P$ or $r \in (P :_R M)$ [11]. A non-zero submodule S of M is said to be *second* if for each $a \in R$, the homomorphism $S \xrightarrow{a} S$ is either surjective or zero [17]. In this case $\text{Ann}_R(S)$ is a prime ideal of R .

The notion of 2-absorbing ideals as a generalization of prime ideals was introduced and studied in [7]. A proper ideal I of R is a *2-absorbing ideal* of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The notion of 2-absorbing ideals was extended to 2-absorbing submodules in [10]. A proper submodule N of M is called a *2-absorbing submodule* of M if whenever $abm \in N$ for some $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In [5], the present authors introduced the dual notion of 2-absorbing submodules (that is, *2-absorbing (resp. strongly 2-absorbing) second submodules*) of M and investigated some properties of these classes of modules. A non-zero submodule N of M is said to be a *2-absorbing second submodule* of M if whenever $a, b \in R$, L is a completely irreducible submodule of M , and $abN \subseteq L$, then $aN \subseteq L$ or $bN \subseteq L$ or $ab \in \text{Ann}_R(N)$. A non-zero submodule N of M is said to be a *strongly 2-absorbing second submodule* of M if whenever $a, b \in R$, K is a submodule of M , and $abN \subseteq K$, then $aN \subseteq K$ or $bN \subseteq K$ or $ab \in \text{Ann}_R(N)$.

In [14], the authors introduced the notion of classical 2-absorbing submodules as a generalization of 2-absorbing submodules and studied some properties of this class of modules. A proper submodule N of M is called *classical 2-absorbing submodule* if whenever $a, b, c \in R$ and $m \in M$ with $abcm \in N$, then $abm \in N$ or $acm \in N$ or $bcm \in N$ [14].

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The purpose of this paper is to introduce the concepts of classical and strongly classical 2-absorbing second submodules of an R -module M as dual notion of classical 2-absorbing submodules and provide some information concerning these new classes of modules. We characterize classical (resp. strongly classical) 2-absorbing second submodules in Theorem 2.3 (resp. Theorem 3.4). Example 3.6 shows that the classes of strongly 2-absorbing second modules and strongly classical 2-absorbing second modules are different in general. Also, we consider the relationship between classical 2-absorbing and strongly classical 2-absorbing second submodules in Examples 3.9, 3.10, and Propositions 3.11. Theorem 2.14 (resp. Theorem 3.15) of this paper shows that if M is an Artinian R -module, then every non-zero submodule of M has only a finite number of maximal classical (resp. strongly classical) 2-absorbing second submodules. Further, among other results, we investigate strongly classical 2-absorbing second submodules of a finite direct product of modules in Theorem 3.19.

2. CLASSICAL 2-ABSORBING SECOND SUBMODULES

Let M be an R -module. A proper submodule N of M is said to be *completely irreducible* if $N = \bigcap_{i \in I} N_i$, where $\{N_i\}_{i \in I}$ is a family of submodules of M , implies that $N = N_i$ for some $i \in I$. It is easy to see that every submodule of M is an intersection of completely irreducible submodules of M [12].

We frequently use the following basic fact without further comment.

Remark 2.1. Let N and K be two submodules of an R -module M . To prove $N \subseteq K$, it is enough to show that if L is a completely irreducible submodule of M such that $K \subseteq L$, then $N \subseteq L$.

Definition 2.2. Let N be a non-zero submodule of an R -module M . We say that N is a *classical 2-absorbing second submodule* of M if whenever $a, b, c \in R$, L is a completely irreducible submodule of M , and $abcN \subseteq L$, then $abN \subseteq L$ or $bcN \subseteq L$ or $acN \subseteq L$. We say M is a *classical 2-absorbing second module* if M is a classical 2-absorbing second submodule of itself.

Theorem 2.3. Let M be an R -module and N be a non-zero submodule of M . Then the following statements are equivalent:

- (a) N is a classical 2-absorbing second submodule of M ;
- (b) For every $a, b \in R$ and completely irreducible submodule L of M with $abN \not\subseteq L$, $(L :_R abN) = (L :_R aN) \cup (L :_R bN)$;
- (c) For every $a, b \in R$ and completely irreducible submodule L of M with $abN \not\subseteq L$, $(L :_R abN) = (L :_R aN)$ or $(L :_R abN) = (L :_R bN)$;
- (d) For every $a, b \in R$, every ideal I of R , and completely irreducible submodule L of M with $abIN \subseteq L$, either $abN \subseteq L$ or $aIN \subseteq L$ or $bIN \subseteq L$;
- (e) For every $a \in R$, every ideal I of R , and completely irreducible submodule L of M with $aIN \not\subseteq L$, $(L :_R aIN) = (L :_R IN)$ or $(L :_R aIN) = (L :_R aN)$;
- (f) For every $a \in R$, ideals I, J of R , and completely irreducible submodule L of M with $aIJN \subseteq L$, either $aIN \subseteq L$ or $aJN \subseteq L$ or $IJN \subseteq L$;
- (g) For ideals I, J of R , and completely irreducible submodule L of M with $IJN \not\subseteq L$, $(L :_R IJN) = (L :_R IN)$ or $(L :_R IJN) = (L :_R JN)$;
- (h) For ideals I_1, I_2, I_3 of R , and completely irreducible submodule L of M with $I_1I_2I_3N \subseteq L$, either $I_1I_2N \subseteq L$ or $I_1I_3N \subseteq L$ or $I_2I_3N \subseteq L$;

- (i) For each completely irreducible submodule L of M with $N \not\subseteq L$, $(L :_R N)$ is a 2-absorbing ideal of R .

Proof. (a) \Rightarrow (b) Let $t \in (L :_R abN)$. Then $tabN \subseteq L$. Since $abN \not\subseteq L$, $atN \subseteq L$ or $btN \subseteq L$ as needed.

(b) \Rightarrow (c) This follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them.

(c) \Rightarrow (d) Let for some $a, b \in R$, an ideal I of R , and completely irreducible submodule L of M , $abIN \subseteq L$. Then $I \subseteq (L :_R abN)$. If $abN \subseteq L$, then we are done. Assume that $abN \not\subseteq L$. Then by part (c), $I \subseteq (L :_R bN)$ or $I \subseteq (L :_R aN)$ as desired.

(d) \Rightarrow (e) \Rightarrow (f) \Rightarrow (g) \Rightarrow (h) The proofs are similar to that of the previous implications.

(h) \Rightarrow (a) Trivial.

(h) \Leftrightarrow (i) This is straightforward. \square

We recall that an R -module M is said to be a *cocyclic module* if $\text{Soc}_R(M)$ is a large and simple submodule of M [18]. (Here $\text{Soc}_R(M)$ denotes the sum of all minimal submodules of M .) A submodule L of M is a completely irreducible submodule of M if and only if M/L is a cocyclic R -module [12].

Corollary 2.4. Let N be a classical 2-absorbing second submodule of a cocyclic R -module M . Then $\text{Ann}_R(N)$ is a 2-absorbing ideal of R .

Proof. This follows from Theorem 2.3 (a) \Rightarrow (i), because (0) is a completely irreducible submodule of M . \square

Example 2.5. For any prime integer p , let $M = \mathbb{Z}_{p^\infty}$ as a \mathbb{Z} -module and $G_i = \langle 1/p^i + \mathbb{Z} \rangle$ for $i = 1, 2, 3, \dots$. Then G_i is not a classical 2-absorbing second submodule of M for $i = 3, 4, 5, \dots$.

Lemma 2.6. Every 2-absorbing second submodule of M is a classical 2-absorbing second submodule of M .

Proof. Let N be a 2-absorbing second submodule of M , $a, b, c \in R$, L a completely irreducible submodule of M , and $abcN \subseteq L$. Then $abN \subseteq (L :_M c)$. Thus $aN \subseteq (L :_M c)$ or $bN \subseteq (L :_M c)$ or $abN = 0$ because by [6, 2.1], $(L :_M c)$ is a completely irreducible submodule of M . Hence $acN \subseteq L$ or $bcN \subseteq L$ or $abN \subseteq L$ as needed. \square

Example 2.7. Consider $M = \mathbb{Z}_{pq} \oplus \mathbb{Q}$ as a \mathbb{Z} -module, where p, q are prime integers. Then M is a classical 2-absorbing second module which is not a strongly 2-absorbing second module.

Proposition 2.8. Let N be a classical 2-absorbing second submodule of an R -module M . Then we have the following.

- (a) If $a \in R$, then $a^n N = a^{n+1} N$, for all $n \geq 2$.
- (b) If L is a completely irreducible submodule of M such that $N \not\subseteq L$, then $\sqrt{(L :_R N)}$ is a 2-absorbing ideal of R .

Proof. (a) It is enough to show that $a^2 N = a^3 N$. It is clear that $a^3 N \subseteq a^2 N$. Let L be a completely irreducible submodule of M such that $a^3 N \subseteq L$. Since N is a classical 2-absorbing second submodule, $a^2 N \subseteq L$. This implies that $a^2 N \subseteq a^3 N$.

(b) Assume that $a, b, c \in R$ and $abc \in \sqrt{(L :_R N)}$. Then there is a positive integer t such that $a^t b^t c^t N \subseteq L$. By hypotheses, N is a classical 2-absorbing second submodule of M , thus $a^t N \subseteq L$ or $b^t N \subseteq L$ or $c^t N \subseteq L$. Therefore, $a \in \sqrt{(L :_R N)}$ or $b \in \sqrt{(L :_R N)}$ or $c \in \sqrt{(L :_R N)}$. \square

Theorem 2.9. *Let N be a submodule of an R -module M . Then we have the following.*

- (a) *If N is a classical 2-absorbing second submodule of M , then IN is a classical 2-absorbing second submodule of M for all ideals I of R with $I \not\subseteq \text{Ann}_R(N)$.*
- (b) *If N is a classical 2-absorbing submodule of M , then $(N :_R I)$ is a classical 2-absorbing submodule of M for all ideals I of R with $I \not\subseteq (N :_R M)$.*
- (c) *Let $f : M \rightarrow \hat{M}$ be a monomorphism of R -modules. If \hat{N} is a classical 2-absorbing second submodule of $f(M)$, then $f^{-1}(\hat{N})$ is a classical 2-absorbing second submodule of M .*

Proof. (a) Let I be an ideal of R with $I \not\subseteq \text{Ann}_R(N)$, $a, b, c \in R$, L be a completely irreducible submodule of M , and $abcIN \subseteq L$. Then $acN \subseteq L$ or $cbIN \subseteq L$ or $abIN = 0$ by Theorem 2.3(a) \Rightarrow (d). If $cbIN \subseteq L$ or $abIN = 0$, then we are done. If $acN \subseteq L$, then $acIN \subseteq acN$ implies that $acIN \subseteq L$, as needed. Since $I \not\subseteq \text{Ann}_R(N)$, we have IN is a non-zero submodule of M .

(b) Use the technique of part (a) and apply [14, Theorem 2].

(c) If $f^{-1}(\hat{N}) = 0$, then $f(M) \cap \hat{N} = ff^{-1}(\hat{N}) = f(0) = 0$. Thus $\hat{N} = 0$, a contradiction. Therefore, $f^{-1}(\hat{N}) \neq 0$. Now let $a, b, c \in R$, L be a completely irreducible submodule of M , and $abcf^{-1}(\hat{N}) \subseteq L$. Then

$$abc\hat{N} = abc(f(M) \cap \hat{N}) = abcf^{-1}(\hat{N}) \subseteq f(L).$$

By [5, 3.14], $f(L)$ is a completely irreducible submodule of $f(M)$. Thus as \hat{N} is a classical 2-absorbing second submodule, $ab\hat{N} \subseteq f(L)$ or $bc\hat{N} \subseteq f(L)$ or $ac\hat{N} \subseteq f(L)$. Therefore, $abf^{-1}(\hat{N}) \subseteq f^{-1}f(L) = L$ or $bcf^{-1}(\hat{N}) \subseteq f^{-1}f(L) = L$ or $acf^{-1}(\hat{N}) \subseteq f^{-1}f(L) = L$, as desired. \square

An R -module M is said to be a *multiplication module* if for every submodule N of M there exists an ideal I of R such that $N = IM$ [8].

An R -module M is said to be a *comultiplication module* if for every submodule N of M there exists an ideal I of R such that $N = (0 :_M I)$, equivalently, for each submodule N of M , we have $N = (0 :_M \text{Ann}_R(N))$ [2].

Corollary 2.10. *Let M be an R -module. Then we have the following.*

- (a) *If M is a multiplication classical 2-absorbing second R -module, then every non-zero submodule of M is a classical 2-absorbing second submodule of M .*
- (b) *If M is a comultiplication module and the zero submodule of M is a classical 2-absorbing submodule, then every proper submodule of M is a classical 2-absorbing submodule of M .*

Proof. This follows from parts (a) and (b) of Theorem 2.9. \square

Proposition 2.11. *Let M be an R -module and $\{K_i\}_{i \in I}$ be a chain of classical 2-absorbing second submodules of M . Then $\sum_{i \in I} K_i$ is a classical 2-absorbing second submodule of M .*

Proof. Let $a, b, c \in R$, L be a completely irreducible submodule of M , and $abc \sum_{i \in I} K_i \subseteq L$. Assume that $ab \sum_{i \in I} K_i \not\subseteq L$ and $ac \sum_{i \in I} K_i \not\subseteq L$. Then there are $m, n \in I$ where $abK_n \not\subseteq L$ and $acK_m \not\subseteq L$. Hence, for every $K_n \subseteq K_s$ and every $K_m \subseteq K_d$ we have that $abK_s \not\subseteq L$ and $acK_d \not\subseteq L$. Therefore, for each submodule K_h such that $K_n \subseteq K_h$ and $K_m \subseteq K_h$, we have $bcK_h \subseteq L$. Hence $bc \sum_{i \in I} K_i \subseteq L$, as needed. \square

Definition 2.12. We say that a classical 2-absorbing second submodule N of an R -module M is a *maximal classical 2-absorbing second submodule* of a submodule K of M , if $N \subseteq K$ and there does not exist a classical 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 2.13. Let M be an R -module. Then every classical 2-absorbing second submodule of M is contained in a maximal classical 2-absorbing second submodule of M .

Proof. This is proved easily by using Zorn's Lemma and Proposition 2.11. \square

Theorem 2.14. Let M be an Artinian R -module. Then every non-zero submodule of M has only a finite number of maximal classical 2-absorbing second submodules.

Proof. Suppose that there exists a non-zero submodule N of M such that it has an infinite number of maximal classical 2-absorbing second submodules. Let S be a submodule of M chosen minimal such that S has an infinite number of maximal classical 2-absorbing second submodules because M is an Artinian R -module. Then S is not a classical 2-absorbing second submodule. Thus there exist $a, b, c \in R$ and a completely irreducible submodule L of M such that $abcS \subseteq L$ but $abS \not\subseteq L$, $acS \not\subseteq L$, and $bcS \not\subseteq L$. Let V be a maximal classical 2-absorbing second submodule of M contained in S . Then $abV \subseteq L$ or $acV \subseteq L$ or $bcV \subseteq L$. Thus $V \subseteq (L :_M ab)$ or $V \subseteq (L :_M ac)$ or $V \subseteq (L :_M bc)$. Therefore, $V \subseteq (L :_S ab)$ or $V \subseteq (L :_S ac)$ or $V \subseteq (L :_S bc)$. By the choice of S , the modules $(L :_S ab)$, $(L :_S ac)$, and $(L :_S bc)$ have only finitely many maximal classical 2-absorbing second submodules. Therefore, there is only a finite number of possibilities for the module S , which is a contradiction. \square

3. STRONGLY CLASSICAL 2-ABSORBING SECOND SUBMODULES

Definition 3.1. Let N be a non-zero submodule of an R -module M . We say that N is a *strongly classical 2-absorbing second submodule* of M if whenever $a, b, c \in R$, L_1, L_2, L_3 are completely irreducible submodules of M , and $abcN \subseteq L_1 \cap L_2 \cap L_3$, then $abN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$. We say M is a *strongly classical 2-absorbing second module* if M is a strongly classical 2-absorbing second submodule of itself.

Example 3.2. Clearly every strongly classical 2-absorbing second submodule is a classical 2-absorbing second submodule. But the converse is not true in general. For example, consider \mathbb{Z} as a \mathbb{Z} -module. Then $2\mathbb{Z}$ is a classical 2-absorbing second submodule of \mathbb{Z} but it is not a strongly classical 2-absorbing second submodule of \mathbb{Z} .

Example 3.3. The \mathbb{Z} -module \mathbb{Z} has no strongly classical 2-absorbing second submodule.

Theorem 3.4. *Let M be an R -module and N be a non-zero submodule of M . Then the following statements are equivalent:*

- (a) N is strongly classical 2-absorbing second;
- (b) If $a, b, c \in R$, K is a submodule of M , and $abcN \subseteq K$, then $abN \subseteq K$ or $bcN \subseteq K$ or $acN \subseteq K$;
- (c) For every $a, b, c \in R$, $abcN = abN$ or $abcN = acN$ or $abcN = bcN$;
- (d) For every $a, b \in R$ and submodule K of M with $abN \not\subseteq K$, $(K :_R abN) = (K :_R aN) \cup (K :_R bN)$;
- (e) For every $a, b \in R$ and submodule K of M with $abN \not\subseteq K$, $(K :_R abN) = (K :_R aN)$ or $(K :_R abN) = (K :_R bN)$;
- (f) For every $a, b \in R$, every ideal I of R , and submodule K of M with $abIN \subseteq K$, either $abN \subseteq K$ or $aIN \subseteq K$ or $bIN \subseteq K$;
- (g) For every $a \in R$, every ideal I of R , and submodule K of M with $aIN \not\subseteq K$, $(K :_R aIN) = (K :_R IN)$ or $(K :_R aIN) = (K :_R aN)$;
- (h) For every $a \in R$, ideals I, J of R , and submodule K of M with $aIJN \subseteq K$, either $aIN \subseteq K$ or $aJN \subseteq K$ or $IJN \subseteq K$;
- (i) For ideals I, J of R , and submodule K of M with $IJN \not\subseteq K$, $(K :_R IJN) = (K :_R IN)$ or $(K :_R IJN) = (K :_R JN)$;
- (j) For ideals I_1, I_2, I_3 of R , and submodule K of M with $I_1I_2I_3N \subseteq K$, either $I_1I_2N \subseteq K$ or $I_1I_3N \subseteq K$ or $I_2I_3N \subseteq K$;
- (k) For each submodule K of M with $N \not\subseteq K$, $(K :_R N)$ is a 2-absorbing ideal of R .

Proof. (a) \Rightarrow (b) Let $a, b, c \in R$, K be a submodule of M , and $abcN \subseteq K$. Assume on the contrary that $abN \not\subseteq K$, $bcN \not\subseteq K$, and $acN \not\subseteq K$. Then there exist completely irreducible submodules L_1, L_2, L_3 of M such that K is a submodule of them but $abN \not\subseteq L_1$, $bcN \not\subseteq L_2$, and $acN \not\subseteq L_3$. Now we have $abcN \subseteq L_1 \cap L_2 \cap L_3$. Thus by part (a), $abN \subseteq L_1 \cap L_2 \cap L_3$ or $bcN \subseteq L_1 \cap L_2 \cap L_3$ or $acN \subseteq L_1 \cap L_2 \cap L_3$. Therefore, $abN \subseteq L_1$ or $bcN \subseteq L_2$ or $acN \subseteq L_3$ which are contradictions.

(b) \Rightarrow (c) Let $a, b, c \in R$. Then $abcN \subseteq abcN$ implies that $abN \subseteq abcN$ or $bcN \subseteq abcN$ or $acN \subseteq abcN$ by part (b). Thus $abN = abcN$ or $bcN = abcN$ or $acN = abcN$ because the reverse inclusions are clear.

(c) \Rightarrow (d) Let $t \in (K :_R abN)$. Then $tabN \subseteq K$. Since $abN \not\subseteq K$, $atN \subseteq K$ or $btN \subseteq K$ as needed.

(d) \Rightarrow (e) This follows from the fact that if an ideal is the union of two ideals, then it is equal to one of them.

(e) \Rightarrow (f) Let for some $a, b \in R$, an ideal I of R , and submodule K of M , $abIN \subseteq K$. Then $I \subseteq (K :_R abN)$. If $abN \subseteq K$, then we are done. Assume that $abN \not\subseteq K$. Then by part (d), $I \subseteq (K :_R bN)$ or $I \subseteq (K :_R aN)$ as desired.

(g) \Rightarrow (h) \Rightarrow (i) \Rightarrow (h) \Rightarrow (j) Have proofs similar to that of the previous implications.

(j) \Rightarrow (a) Trivial.

(j) \Leftrightarrow (k) This is straightforward. \square

Let N be a submodule of an R -module M . Then Theorem 3.4 (a) \Leftrightarrow (c) shows that N is a strongly classical 2-absorbing second submodule of M if and only if N is a strongly classical 2-absorbing second module.

Corollary 3.5. Let N be a strongly classical 2-absorbing second submodule of an R -module M and I be an ideal of R . Then $I^n N = I^{n+1} N$, for all $n \geq 2$.

Proof. It is enough to show that $I^2N = I^3N$. By Theorem 3.4, $I^2N = I^3N$. \square

Example 3.6. Clearly every strongly 2-absorbing second submodule is a strongly classical 2-absorbing second submodule. But the converse is not true in general. For example, consider $M = \mathbb{Z}_6 \oplus \mathbb{Q}$ as a \mathbb{Z} -module. Then M is a strongly classical 2-absorbing second module. But M is not a strongly 2-absorbing second module.

A non-zero submodule N of an R -module M is said to be a *weakly second submodule* of M if $rsN \subseteq K$, where $r, s \in R$ and K is a submodule of M , implies either $rN \subseteq K$ or $sN \subseteq K$ [1].

Proposition 3.7. Let M be an R -module. Then we have the following.

- (a) If M is a comultiplication R -module and N is a strongly classical 2-absorbing second submodule of M , then N is a strongly 2-absorbing second submodule of M .
- (b) If N_1, N_2 are weakly second submodules of M , then $N_1 + N_2$ is a strongly classical 2-absorbing second submodule of M .
- (c) If N is a strongly classical 2-absorbing second submodule of M , then IN is a strongly classical 2-absorbing second submodule of M for all ideals I of R with $I \not\subseteq \text{Ann}_R(N)$.
- (d) If M is a multiplication strongly classical 2-absorbing second R -module, then every non-zero submodule of M is a classical 2-absorbing second submodule of M .
- (e) If M is a strongly classical 2-absorbing second R -module, then every non-zero homomorphic image of M is a classical 2-absorbing second R -module.

Proof. (a) By Theorem 3.4 (a) \Rightarrow (k), $\text{Ann}_R(N)$ is a 2-absorbing ideal of R . Now the result follows from [5, 3.10].

(b) Let N_1, N_2 be weakly second submodules of M and $a, b, c \in R$. Since N_1 is a weakly second submodule, we may assume that $abcN_1 = aN_1$. Likewise, assume that $abcN_2 = bN_2$. Hence $abc(N_1 + N_2) = ab(N_1 + N_2)$ which implies $N_1 + N_2$ is a classical 2-absorbing second submodule by Theorem 3.4 (c) \Rightarrow (a).

(c) Use the technique of the proof of Theorem 2.9 (a).

(d) This follows from part (c).

(e) This is straightforward. \square

For a submodule N of an R -module M the *second radical* (or *second socle*) of N is defined as the sum of all second submodules of M contained in N and it is denoted by $\text{sec}(N)$ (or $\text{soc}(N)$). In case N does not contain any second submodule, the second radical of N is defined to be (0) (see [9] and [3]).

Theorem 3.8. Let M be a finitely generated comultiplication R -module. If N is a strongly classical 2-absorbing second submodule of M , then $\text{sec}(N)$ is a strongly 2-absorbing second submodule of M .

Proof. Let N be a strongly classical 2-absorbing second submodule of M . By Proposition 3.7 (a), $\text{Ann}_R(N)$ is a 2-absorbing ideal of R . Thus by [7, 2.1], $\sqrt{\text{Ann}_R(N)}$ is a 2-absorbing ideal of R . By [4, 2.12], $\text{Ann}_R(\text{sec}(N)) = \sqrt{\text{Ann}_R(N)}$. Therefore, $\text{Ann}_R(\text{sec}(N))$ is a 2-absorbing ideal of R . Now the result follows from [5, 3.10]. \square

The following examples show that the two concepts of classical 2-absorbing submodules and strongly classical 2-absorbing second submodules are different in general.

Example 3.9. The submodule $2\mathbb{Z}$ of the \mathbb{Z} -module \mathbb{Z} is a classical 2-absorbing submodule which is not a strongly classical 2-absorbing second module.

Example 3.10. The submodule $\langle 1/p + \mathbb{Z} \rangle$ of the \mathbb{Z} -module \mathbb{Z}_{p^∞} is a strongly classical 2-absorbing second module which is not a classical 2-absorbing submodule of \mathbb{Z}_{p^∞} .

A commutative ring R is said to be a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R -module which is equal to a finite union of submodules must be equal to one of them [15].

In the following proposition, we investigate the relationships between strongly classical 2-absorbing second submodules and classical 2-absorbing submodules.

Proposition 3.11. Let M be a non-zero R -module. Then we have the following.

- (a) If M is a finitely generated strongly classical 2-absorbing second R -module, then the zero submodule of M is a classical 2-absorbing submodule.
- (b) If M is a multiplication strongly classical 2-absorbing second R -module, then the zero submodule of M is a classical 2-absorbing submodule.
- (c) Let R be a *um-ring*. If M is an Artinian R -module and the zero submodule of M is a classical 2-absorbing submodule, then M is a strongly classical 2-absorbing second R -module.
- (d) Let R be a *um-ring*. If M is a comultiplication R -module and the zero submodule of M is a classical 2-absorbing submodule, then M is a strongly classical 2-absorbing second R -module.

Proof. (a) Let $a, b, c \in R$, $m \in M$, and $abcm = 0$. By Theorem 3.4, we can assume that $abcM = acM$. Since M is finitely generated, by using [13, Theorem 76], $\text{Ann}_R(abM) + Rc = R$. It follows that $(0 :_M abc) = (0 :_M ab)$. This implies that $abm = 0$, as needed.

(b) Let $a, b, c \in R$, $m \in M$, and $abcm = 0$. Then by Theorem 3.4, we can assume that $abcM = acM$. Thus

$$0 = abc((0 :_M abc) :_R M)M = (((0 :_M abc) :_R M)M)ab.$$

Since M is a multiplication module, $((0 :_M abc) :_R M)M = (0 :_M abc)$. Therefore, $(0 :_M abc)ab = 0$. It follows that $(0 :_M abc) \subseteq (0 :_M ab)$. Thus $(0 :_M abc) = (0 :_M ab)$ because the reverse inclusion is clear. Hence $abm = 0$, as required.

(c) Let $a, b, c \in R$. Then by [14, Theorem 4], we can assume that $(0 :_M abc) = (0 :_M ab)$. Hence $(0 :_{M/(0 :_M ab)} c) = 0$. Since M is Artinian, it follows that $cM + (0 :_M ab) = M$. Therefore, $abcM = abM$. Thus by Theorem 3.4 (c) \Rightarrow (a), M is a classical 2-absorbing second R -module.

(d) Let $a, b, c \in R$. Then by [14, Theorem 4], we can assume that $(0 :_M abc) = (0 :_M ab)$. Since M is a comultiplication R -module, this implies that

$$M = ((0 :_M abc) :_M \text{Ann}_R(abcM)) = ((0 :_M ab) :_M \text{Ann}_R(abcM)) = (abcM :_M ab).$$

It follows that $abM \subseteq abcM$. Thus $abM = abcM$ because the reverse implication is clear and this completed the proof. \square

Proposition 3.12. Let M be an R -module and $\{K_i\}_{i \in I}$ be a chain of strongly classical 2-absorbing second submodules of M . Then $\sum_{i \in I} K_i$ is a strongly classical 2-absorbing second submodule of M .

Proof. Use the technique of Proposition 2.11. \square

Definition 3.13. We say that a strongly classical 2-absorbing second submodule N of an R -module M is a *maximal strongly classical 2-absorbing second submodule* of a submodule K of M , if $N \subseteq K$ and there does not exist a strongly classical 2-absorbing second submodule T of M such that $N \subset T \subset K$.

Lemma 3.14. Let M be an R -module. Then every strongly classical 2-absorbing second submodule of M is contained in a maximal strongly classical 2-absorbing second submodule of M .

Proof. This is proved easily by using Zorn's Lemma and Proposition 3.12. \square

Theorem 3.15. Let M be an Artinian R -module. Then every non-zero submodule of M has only a finite number of maximal strongly classical 2-absorbing second submodules.

Proof. Use the technique of Theorem 2.14 any apply Lemma 3.14. \square

Theorem 3.16. Let $f : M \rightarrow \acute{M}$ be a monomorphism of R -modules. Then we have the following.

- (a) If N is a strongly classical 2-absorbing second submodule of M , then $f(N)$ is a strongly classical 2-absorbing second submodule of \acute{M} .
- (b) If \acute{N} is a strongly classical 2-absorbing second submodule of $f(M)$, then $f^{-1}(\acute{N})$ is a strongly classical 2-absorbing second submodule of M .

Proof. (a) Since $N \neq 0$ and f is a monomorphism, we have $f(N) \neq 0$. Let $a, b, c \in R$. Then by Theorem 3.4 (a) \Rightarrow (c), we can assume that $abcN = abN$. Thus

$$abcf(N) = f(abcN) = f(abN) = abf(N).$$

Hence $f(N)$ is a classical 2-absorbing second submodule of \acute{M} by Theorem 3.4 (c) \Rightarrow (a).

(b) If $f^{-1}(\acute{N}) = 0$, then $f(M) \cap \acute{N} = ff^{-1}(\acute{N}) = f(0) = 0$. Thus $\acute{N} = 0$, a contradiction. Therefore, $f^{-1}(\acute{N}) \neq 0$. Now let $a, b, c \in R$, K be a submodule of M , and $abcf^{-1}(\acute{N}) \subseteq K$. Then

$$abc\acute{N} = abc(f(M) \cap \acute{N}) = abcf f^{-1}(\acute{N}) \subseteq f(K).$$

Thus as \acute{N} is a strongly classical 2-absorbing second submodule, $ab\acute{N} \subseteq f(K)$ or $bc\acute{N} \subseteq f(K)$ or $ac\acute{N} \subseteq f(K)$. Therefore, $abf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $bcf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$ or $acf^{-1}(\acute{N}) \subseteq f^{-1}f(K) = K$, as desired. \square

Let R_i be a commutative ring with identity and M_i be an R_i -module for $i = 1, 2$. Let $R = R_1 \times R_2$. Then $M = M_1 \times M_2$ is an R -module and each submodule of M is in the form of $N = N_1 \times N_2$ for some submodules N_1 of M_1 and N_2 of M_2 .

Theorem 3.17. Let $R = R_1 \times R_2$ be a decomposable ring and let $M = M_1 \times M_2$ be an R -module, where M_1 is an R_1 -module and M_2 is an R_2 -module. Suppose that $N = N_1 \times N_2$ is a non-zero submodule of M . Then the following conditions are equivalent:

- (a) N is a strongly classical 2-absorbing second submodule of M ;
- (b) Either $N_1 = 0$ and N_2 is a strongly classical 2-absorbing second submodule of M_2 or $N_2 = 0$ and N_1 is a strongly classical 2-absorbing second submodule of M_1 or N_1, N_2 are weakly second submodules of M_1, M_2 , respectively.

Proof. (a) \Rightarrow (b). Suppose that N is a strongly classical 2-absorbing second submodule of M such that $N_2 = 0$. From our hypothesis, N is non-zero, so $N_1 \neq 0$. Set $\dot{M} = M_1 \times 0$. One can see that $\dot{N} = N_1 \times 0$ is a strongly classical 2-absorbing second submodule of \dot{M} . Also observe that $\dot{M} \cong M_1$ and $\dot{N} \cong N_1$. Thus N_1 is a strongly classical 2-absorbing second submodule of M_1 . Suppose that $N_1 \neq 0$ and $N_2 \neq 0$. We show that N_1 is a weakly second submodule of M_1 . Since $N_2 \neq 0$, there exists a completely irreducible submodule L_2 of M_2 such that $N_2 \not\subseteq L_2$. Let $abN_1 \subseteq K$ for some $a, b \in R_1$ and submodule K of M_1 . Thus $(a, 1)(b, 1)(1, 0)(N_1 \times N_2) = abN_1 \times 0 \subseteq K \times L_2$. So either $(a, 1)(b, 1)(N_1 \times N_2) = abN_1 \times N_2 \subseteq K \times L_2$ or $(a, 1)(1, 0)(N_1 \times N_2) = aN_1 \times 0 \subseteq K \times L_2$ or $(b, 1)(1, 0)(N_1 \times N_2) = bN_1 \times 0 \subseteq K \times L_2$. If $abN_1 \times N_2 \subseteq K \times L_2$, then $N_2 \subseteq L_2$, a contradiction. Hence either $aN_1 \subseteq K$ or $bN_1 \subseteq K$ which shows that N_1 is a weakly second submodule of M_1 . Similarly, we can show that N_2 is a weakly second submodule of M_2 .

(b) \Rightarrow (a). Suppose that $N = N_1 \times 0$, where N_1 is a strongly classical 2-absorbing (resp. weakly) second submodule of M_1 . Then it is clear that N is a strongly classical 2-absorbing (resp. weakly) second submodule of M . Now, assume that $N = N_1 \times N_2$, where N_1 and N_2 are weakly second submodules of M_1 and M_2 , respectively. Hence $(N_1 \times 0) + (0 \times N_2) = N_1 \times N_2 = N$ is a strongly classical 2-absorbing second submodule of M , by Proposition 3.7 (b). \square

Lemma 3.18. Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a decomposable ring and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R -module where for every $1 \leq i \leq n$, M_i is an R_i -module, respectively. A non-zero submodule N of M is a weakly second submodule of M if and only if $N = \times_{i=1}^n N_i$ such that for some $k \in \{1, 2, \dots, n\}$, N_k is a weakly second submodule of M_k , and $N_i = 0$ for every $i \in \{1, 2, \dots, n\} \setminus \{k\}$.

Proof. (\Rightarrow) Let N be a weakly second submodule of M . We know $N = \times_{i=1}^n N_i$ where for every $1 \leq i \leq n$, N_i is a submodule of M_i , respectively. Assume that N_r is a non-zero submodule of M_r and N_s is a non-zero submodule of M_s for some $1 \leq r < s \leq n$. Since N is a weakly second submodule of M ,

$$(0, \dots, 0, 1_{R_r}, 0, \dots, 0)(0, \dots, 0, 1_{R_s}, 0, \dots, 0)N = (0, \dots, 0, 1_{R_r}, 0, \dots, 0)N$$

or

$$(0, \dots, 0, 1_{R_r}, 0, \dots, 0)(0, \dots, 0, 1_{R_s}, 0, \dots, 0)N = (0, \dots, 0, 1_{R_s}, 0, \dots, 0)N.$$

Thus $N_r = 0$ or $N_s = 0$. This contradiction shows that exactly one of the N_i s is non-zero, say N_k . Now, we show that N_k is a weakly second submodule of M_k . Let $a, b \in R_k$. Since N is a weakly second submodule of M ,

$$(0, \dots, 0, a, 0, \dots, 0)(0, \dots, 0, b, 0, \dots, 0)N = (0, \dots, 0, a, 0, \dots, 0)N$$

or

$$(0, \dots, 0, a, 0, \dots, 0)(0, \dots, 0, b, 0, \dots, 0)N = (0, \dots, 0, b, 0, \dots, 0)N$$

Thus $abN_k = aN_k$ or $abN_k = bN_k$ as needed.

(\Leftarrow) This is clear. \square

Theorem 3.19. Let $R = R_1 \times R_2 \times \cdots \times R_n$ ($2 \leq n < \infty$) be a decomposable ring and $M = M_1 \times M_2 \times \cdots \times M_n$ be an R -module, where for every $1 \leq i \leq n$, M_i is an R_i -module, respectively. Then for a non-zero submodule N of M the following conditions are equivalent:

- (a) N is a strongly classical 2-absorbing second submodule of M ;

- (b) Either $N = \times_{i=1}^n N_i$ such that for some $k \in \{1, 2, \dots, n\}$, N_k is a strongly classical 2-absorbing second submodule of M_k , and $N_i = 0$ for every $i \in \{1, 2, \dots, n\} \setminus \{k\}$ or $N = \times_{i=1}^n N_i$ such that for some $k, m \in \{1, 2, \dots, n\}$, N_k is a weakly second submodule of M_k , N_m is a weakly second submodule of M_m , and $N_i = 0$ for every $i \in \{1, 2, \dots, n\} \setminus \{k, m\}$.

Proof. We use induction on n . For $n = 2$ the result holds by Theorem 3.17. Now let $3 \leq n < \infty$ and suppose that the result is valid when $K = M_1 \times \dots \times M_{n-1}$. We show that the result holds when $M = K \times M_n$. By Theorem 3.17, N is a strongly classical 2-absorbing second submodule of M if and only if either $N = L \times 0$ for some strongly classical 2-absorbing second submodule L of K or $N = 0 \times L_n$ for some strongly classical 2-absorbing second submodule L_n of M_n or $N = L \times L_n$ for some weakly second submodule L of K and some weakly second submodule L_n of M_n . Note that by Lemma 3.18, a non-zero submodule L of K is a weakly second submodule of K if and only if $L = \times_{i=1}^{n-1} N_i$ such that for some $k \in \{1, 2, \dots, n-1\}$, N_k is a weakly second submodule of M_k and $N_i = 0$ for every $i \in \{1, 2, \dots, n-1\} \setminus \{k\}$. Hence the claim is proved. \square

Example 3.20. Let R be a Noetherian ring and let $E = \bigoplus_{m \in \text{Max}(R)} E(R/m)$. Then for each 2-absorbing ideal P of R , $(0 :_E P)$ is a strongly classical 2-absorbing second submodule of E .

Proof. By using [16, p. 147], $\text{Hom}_R(R/P, E) \neq 0$. Now since $(0 :_E P) \cong \text{Hom}_R(R/P, E)$, $(0 :_E P)$ is a strongly 2-absorbing second submodule of E by [5, 3.27]. Now the result follows from Example 3.6. \square

Theorem 3.21. Let R be a um-ring and M be an R -module. If E is an injective R -module and N is a classical 2-absorbing submodule of M such that $\text{Hom}_R(M/N, E) \neq 0$, then $\text{Hom}_R(M/N, E)$ is a strongly classical 2-absorbing second R -module.

Proof. Let $a, b, c \in R$. Since N is a classical 2-absorbing submodule of M , we can assume that $(N :_M abc) = (N :_M ab)$ by [14, Theorem 4]. Since E is an injective R -module, by replacing M with M/N in [1, 3.13 (a)], we have $\text{Hom}_R(M/(N :_M r), E) = r\text{Hom}_R(M/N, E)$ for each $r \in R$. Therefore,

$$abc\text{Hom}_R(M/N, E) = \text{Hom}_R(M/(N :_M abc), E) =$$

$$\text{Hom}_R(M/(N :_M ab), E) = ab\text{Hom}_R(M/N, E),$$

as needed \square

Theorem 3.22. Let M be a strongly classical 2-absorbing second R -module and F be a right exact linear covariant functor over the category of R -modules. Then $F(M)$ is a strongly classical 2-absorbing second R -module if $F(M) \neq 0$.

Proof. This follows from [1, 3.14] and Theorem 3.4 (a) \Rightarrow (c). \square

Corollary 3.23. Let M be an R -module, S be a multiplicative subset of R and N be a strongly classical 2-absorbing second submodule of M . Then $S^{-1}N$ is a strongly classical 2-absorbing second submodule of $S^{-1}M$ if $S^{-1}N \neq 0$.

Proof. This follows from Theorem 3.22. \square

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